SPREADING OF AN ELLIPTICAL FILM OF LIQUID UNDER IMPACT

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The article deals with the impact on an elliptical film of liquid, with the effect of the film shape on the punch deceleration and on the velocity of liquid spreading, and with the stability of the process. The problem is solved for the case of a slowly flowing viscous liquid as well as for an ideal liquid, but with the inertia forces taken into account. The theory is found to agree with experimental data. The inertia-free spreading of a circular film of viscous liquid has already been studied by Reynolds [1]. Interest in this problem is being shown again in connection with certain aspects of lubrication, of the punch and die operation, and with the sensitivity analysis of liquid explosives [2, 3, 4, 5, 6, 7]. Several articles deal with the impact on a circular film [3, 4, 5, 6].

1. Formulation of the Problem. We will consider an axially oriented impact at velocity w_0 on a liquid film having the shape of an elliptical cylinder (height h_0, semiaxes $a_0 \geq b_0$) or of a $2b_0$-wide strip into which an ellipse degenerates when $a_0 \gg b_0$. The compressibility of the liquid as well as of the punch and the die will be disregarded. The ratio h_0/b_0 will be assumed small. With the characteristic scale factors a_0, b_0, h_0 for the dimensions along the x, y, z axes respectively, $h_0/|w_0|$ for time, $\rho w_0^2 b_0^2/h_0^2$ for the pressure, $|w_0|$ for the axial component of velocity, and $|w_0| b_0/h_0$ for the velocities in the x and the y direction, the system of hydrodynamic equations will, after simplification based on the smallness of $(h_0/b_0)^2$, be written as follows:

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2}, \quad \frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2}, \quad \frac{\partial p}{\partial z} = 0$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}, \quad \nu = \frac{\mu}{\rho} \tag{1.1}$$

The boundary conditions here follow from the requirement that the axial velocity of the liquid at the punch and at the die be the same,

$$w|_{z=0} = 0, \quad w|_{z=h} = w^\circ(t), \; h = h_0 + \int_0^t w^\circ dt, \; w^\circ(0) = w_0^\circ \tag{1.2}$$

(w_0° is the initial squeezing velocity) and from the symmetry at the center

$$u(0, 0, t) = 0, \qquad v(0, 0, t) = 0$$

For a viscous liquid, adhesion $u=0$ and $v=0$ should also occur at $z=0$ and $z=h(t)$.

Pressure on the lateral surface of the film will be disregarded.

A solution will be sought in the form [8]

$$u = xU(z, t), \; v = yV(z, t), \; p = P(t)\left[1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right] \tag{1.3}$$

Equations (1.1) will then be satisfied, if

$$a^2\left[\frac{\partial U}{\partial t} + U^2 + w\frac{\partial U}{\partial z} - \nu\frac{\partial^2 U}{\partial z^2}\right] = b^2\left[\frac{\partial V}{\partial t} + V^2 + w\frac{\partial V}{\partial z} - \nu\frac{\partial^2 V}{\partial z^2}\right] \tag{1.4}$$

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$$U+V+\frac{\partial w}{\partial z}=0, \quad P(t)=\frac{a^2\rho}{2}\left[U^2+\frac{\partial U}{\partial t}+w\frac{\partial U}{\partial z}-\nu\frac{\partial^2 U}{\partial z^2}\right]$$

2. Spreading of a Film of Ideal Liquid. We will first consider the case where the inertia forces prevail over the viscous forces, and the liquid may be treated as an ideal one by letting $\nu=0$.

It will be assumed that U and V do not depend on z. We then find from (1.4)

$$w=zW(t), \qquad W=w^\circ(t)/h(t) \tag{2.1}$$

From (1.4) we obtain the equations for U and V,

$$a^2(U^2+U^\cdot)=b^2(V^2+V^\cdot), \quad U+V+W=0$$
$$p=\frac{\rho a^2}{2}(U^2+U^\cdot)\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right) \tag{2.2}$$

as well as for the punch velocity

$$m_0\dot{w}^\circ(t)=\frac{\pi a^3 b\rho}{4}(U^2+U^\cdot) \tag{2.3}$$

The dot signifies a derivative with respect to time.

The initial conditions for these equations can be determined by assuming that during the impact ($0\le t\le t_0$) the accelerations are infinite while the velocities tend toward their finite initial values [5]. Integrating the pressure over the area of contact with the liquid and over the time from 0 to t_0, and then approaching the limit $t_0\to 0$, we find the impulse supplied by the punch equal to the momentum lost by it

$$\Delta I=\lim_{t_0\to 0}\int_0^{t_0}\int_s p\,ds\,dt=\frac{\pi a_0^3 b_0\rho}{4}U_0, \quad \Delta I=m_0(w_0^\circ-w_0)$$

as a quantity independent of the coordinate system so that by interchanging the x and y axes we can write

$$\Delta I=\frac{\pi b_0^3 a_0\rho}{4}U_0, \qquad \text{т. е. } a_0^2U_0=b_0^2V_0 \tag{2.4}$$

From (2.2), (2.3), and (2.4) we find

$$U_0=-\frac{b_0^2W_0}{a_0^2+b_0^2}, \quad V_0=-\frac{a_0^2W_0}{a_0^2+b_0^2}, \quad W_0=\frac{w_0^\circ}{h_0}$$
$$w_0^\circ=\frac{m_0w_0}{m_0+m_*}, \qquad m_*=\frac{\pi\rho a_0^2b_0^3}{4h_0(a_0^2+b_0^2)} \tag{2.5}$$

where m_* is the associated mass, and subscript 0 denotes the initial value of a variable.

The initial conditions (2.5) follow also from the hydrodynamic theory of impact [9, 10, 11].

We will verify this on the simple case where $a_0\to\infty$ and the ellipse degenerates into a $2b_0$-wide strip. The particular solution for the velocities, for the potential Φ_0, and for the specific impulse i_0 then becomes

$$v=-w_0^\circ\frac{y}{h}, \quad w=w_0^\circ\frac{z}{h}, \quad i_0=-\frac{2}{3}\frac{\rho_0w_0^\circ b_0^3}{h_0}$$
$$\Phi_0=w_0^\circ\frac{(z^2-h^2)-(y^2-b^2)}{2h_0} \tag{2.6}$$

In a more rigorous formulation of the problem, the potential of the initial velocity field is determined by solving the Laplace equation $\nabla\Phi=0$ for the conditions

$$\Phi(\pm b_0, z)=0, \quad \partial\Phi/\partial z|_{z=0}=0, \quad \partial\Phi/\partial z|_{z=h_0}=w_0^\circ$$

Let us represent Φ as the sum $\Phi=\Phi_0+\Phi_1$ so that

$$\Delta\Phi_1=0, \quad \Phi_1(\pm b_0, z)=w_0^\circ\frac{h_0^2-z^2}{2h_0}, \quad \partial\Phi_1/\partial z|_{z=0,\,h_0}=0$$

Application of the Fourier method yields here

$$\Phi=w_0^\circ\frac{z^2}{2h_0}+w_0^\circ\frac{b_0^2-y^2}{2h_0}-w_0^\circ\frac{h_0}{6}+\sum_{k=1}^{\infty}\frac{2(-1)^{k+1}w_0^\circ h_0\,\mathrm{ch}(\pi ky/h_0)}{\pi^2k^2\,\mathrm{ch}(\pi b_0k/h_0)}\cos\frac{\pi kz}{h_0} \tag{2.7}$$

It follows from (2.7) that

$$\bar{u} = \frac{1}{h_0}\int_0^{h_0} \frac{\partial \Phi}{\partial y}\, dz = -w_0^\circ \frac{y}{h_0}, \quad i_0 = -2\rho_0 \int_0^{b_0} \Phi(h_0, y)\, dy =$$

$$= -\frac{2\rho_0 w_0^\circ b_0^3}{3h_0}\left[1 - \frac{1}{2}\left(\frac{h_0}{b_0}\right)^2 - \sigma\left(\frac{h_0}{b_0}\right)^3\right], \quad \sigma = \frac{6}{\pi^3}\sum_{k=1}^{\infty}\frac{1}{k^3}\,\mathrm{th}\,\frac{\pi k b_0}{h_0} \tag{2.8}$$

Since $\tanh(\pi k b_0/h_0) \le 1$ and $k^{-3} \le k^{-2}$, then

$$\sigma < \frac{6}{\pi^3}\sum_{k=1}^{\infty} k^{-2} = \frac{1}{\pi}$$

A comparison between (2.6) and (2.8) will show that the particular solution describes the mean-over-the-thickness velocity field in the film, while the initial squeezing velocites $w_0^\circ = w_0 + \Delta I/m_0$ and the associated mass $m_* = -\Delta I/w_0^\circ$ differ from those in the exact solution by an amount of the order of $(h_0/b_0)^2$. It is to be noted, however, that the lateral velocity $u = \partial\Phi/\partial y$ at the punch wall is initially infinite. The formation of a high-velocity shroud when $t_0 \to 0$ is characteristic of many situations and is observed, for example, when a plate strikes the water surface. This phenomenon is qualitatively explained by lateral load relieving waves [5].

The transient time t_* corresponding to the particular solution can be estimated from the condition of potential rise to a maximum difference $(\Phi - \Phi_0)_{max} = w_0^\circ h_0/2$ at the lateral surface of the liquid film, which indicates that the initial conditions (2.6) are not stringent. With the aid of the Lagrange–Cauchy integral we find

$$(\Phi - \Phi_0)_{\max} \approx \int_0^{t_*} \frac{v^2 + w^2}{2}\, dt \approx \frac{1}{2}\left(\frac{w_0^\circ b_0}{h_0}\right)^2 t_*, \quad t_* \approx \frac{h_0}{|w_0^\circ|}\left(\frac{h_0}{b_0}\right)^2$$

Introducing now the following dimensionless functions

$$\theta = \frac{W}{W_0}, \quad \psi = -\frac{2U}{W}, \quad \xi = \frac{h}{h_0}, \quad \tau = \frac{t\,|w_0^\circ|}{h_0}$$

into (2.2) and (2.3), we arrive at a successively integrable system of equations

$$2\xi(1 + \beta\xi)\frac{d\psi}{d\xi} = -\psi(1-\psi)(2-\psi) + \beta\xi\,[\delta\psi^2 + 2(1+\varepsilon)\psi - 4\varepsilon], \quad \psi(1) = \varepsilon \tag{2.9}$$

$$\frac{2\xi(1+\beta)}{\theta}\frac{d\theta}{d\xi} = \psi^2 - 2\psi + 2(1 - \beta\xi), \qquad \theta(1) = 1 \tag{2.10}$$

$$\tau = -\int_1^{\xi}\frac{d\xi}{\xi\theta}, \quad \varepsilon = \frac{2b^2}{a^2+b^2}, \quad \delta = 1 - \varepsilon, \quad \beta = \frac{4h_0 m_0 (a^2+b^2)}{\pi a^3 b^3 \rho} \tag{2.11}$$

If the punch face conforms exactly to a plane film section, then $a = a_0$, $b = b_0$, $\beta = m_0/m_*$, and the equations make up a closed system.

An Abel equation like (2.9) has three singular points in the $\psi \ge 0$ and $0 \le \xi \le 1$ region: saddle point $\xi = 0$, $\psi = 0$; saddle point $\xi = 0$, $\psi = 2$; and node $\xi = 0$, $\psi = 1$, with the slope of the particular solution $d\psi/d\xi = 3\beta(1-\varepsilon)$. Near the node point the integral curves are tangent to the ψ axis with an infinite derivative

$$\psi = 1 + K\xi^{1/2} + 3(1-\varepsilon)\,\xi, \qquad K = \mathrm{const} \tag{2.12}$$

It follows from (2.10) and (2.12) that near the node which corresponds to complete squeezing of the liquid

$$h \to 0, \;\; \theta = B\xi^{1/2}, \;\; B = \mathrm{const}$$

and, consequently, we conclude from (2.11) that the time of punch motion is $\tau \sim \xi^{-1/2} \to \infty$. With the x and y axes interchanged according to the conditions of the problem, function U becomes function V and parameter ε becomes $2-\varepsilon$. With the aid of (2.2), this property can be expressed in dimensionless form,

$$\varphi(\xi, \beta, \varepsilon) = -\varphi(\xi, \beta, 2-\varepsilon), \qquad \varphi = 1 - \psi$$

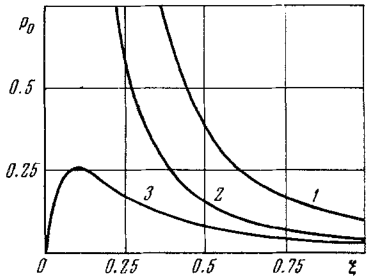

Fig. 1

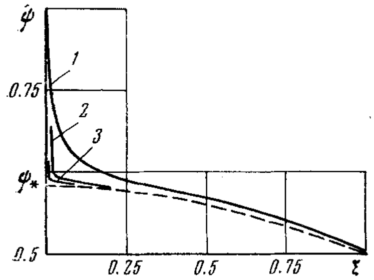

Fig. 2

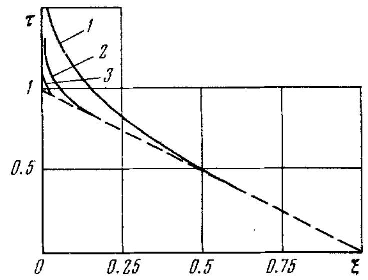

Fig. 3

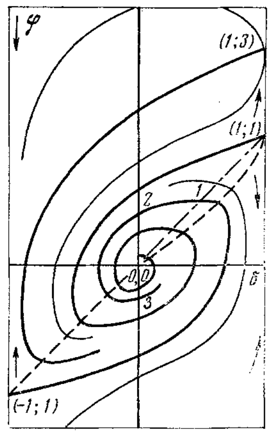

Fig. 4

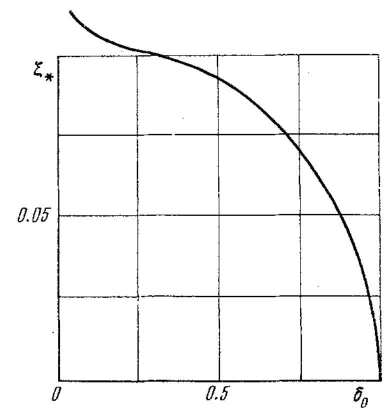

Fig. 5

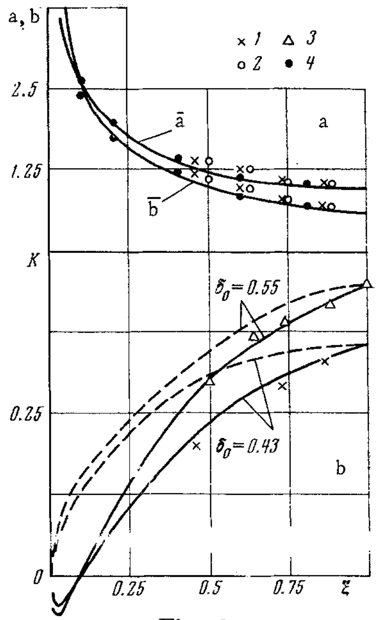

Fig. 6

In the special case where the ellipse degenerates into a circle

$$\varepsilon = 1, \ \varphi(\xi, \beta, 1) = -\varphi(\xi, \beta, 1), \text{i.e.,} \quad \psi = 1$$

Then (2.10) and (2.11) yield

$$\theta = \xi^{1/2}\left(\frac{1+\beta}{1+\beta\xi}\right)^{1/2}, \qquad \tau = \frac{\beta^{1/2}}{(1+\beta)^{3/2}}\left[X(k) - X(k_0)\right] \tag{2.13}$$

$$X(k) = \frac{k(k^2-2)}{k^2-1} - \frac{1}{2}\ln\frac{k+1}{k-1}, \quad k = (1 + 1/\beta\xi)^{1/2}, \quad k_0 = (1 + 1/\beta)^{1/2}$$

The radial velocity and the pressure are simply

$$v_r = -\frac{w_0^\circ}{2h_0} r\theta(\xi), \quad p = \frac{3\beta\rho(1+\beta)^3 (w_0^\circ r_0 \xi)^2}{8h_0^2(1+\beta\xi)^4}\left[1-\left(\frac{r}{r_0}\right)^2\right] \qquad (2.14)$$
$$\beta = \frac{8h_0 q}{\rho r_0^2}, \quad q_0 = \frac{m_0}{\pi r_0^2}$$

It is evident from (2.13) and (2.14) that the spreading velocity $v_r(r_0, t)$ and the pressure at the center $p(0, t)$ reach their maximum values not at once but after some interval of time (see, e.g., curve 3 in Fig. 1, where the change in pressure at the center of a circle $p_0 = Ph_0^2/\rho r_0^2 w_0^2$ is shown for $\beta = 10$). A maximum exists here as a result of two opposing effects: increase in the film thickness and deceleration of the punch.

In the other extreme case $\varepsilon \to 0$, where the ellipse degenerates into a $2b_0$-wide strip, Eqs. (2.9), (2.10), and (2.11) are also easily integrable,

$$\psi \equiv 0, \ \theta = \frac{\xi(1+\beta)^2}{(1+\beta\xi)^2}, \quad \tau = \frac{\beta}{(1+\beta)^2}\left[\frac{1}{\beta\xi} - \frac{1}{\beta} + \beta(1-\xi) - 2\ln\xi\right]$$
$$v = -\frac{w_0^\circ}{h_0} y\theta, \quad p = \frac{\beta\rho\xi}{1+\beta\xi}\left(\frac{w_0^\circ b_0 \theta}{h_0}\right)^2\left[1-\left(\frac{y}{b_0}\right)^2\right] \qquad (2.15)$$
$$\beta = \frac{4h_0 q}{\rho b_0^2}, \quad w_0^\circ = \frac{\beta w_0}{1+\beta}, \quad q = \lim \frac{m_0}{\pi a_0 b_0} \quad (a_0 \to \infty)$$

It is evident from (2.15) that for a finite β the squeezing time becomes

$$t \sim (1+\beta)^{-2}\xi^{-1} \to \infty$$

From (2.2), (2.9), and (2.10) we derive an expression for the pressure at the center of the ellipse,

$$P(\xi) = \frac{\beta\rho\xi}{8(1+\beta\xi)}\left(\frac{w_0^\circ b_0 \theta}{h_0}\right)^2 [(2-\psi)(4-\psi) + \delta\psi^2 + 2(1+\varepsilon)\psi - 4\varepsilon]$$
$$v = -\frac{w_0^\circ}{2h_0} y\theta(2-\psi) \qquad (2.16)$$

A decrease in ε, i.e., a greater eccentricity, reduces the severity of the impact, and with $r_0^2 = a_0 b_0$ the maximum pressure on an ellipse $\psi < 1$ or on a strip is smaller than on a circle $\psi \equiv 1$. This is naturally so since the perimeter of an ellipse is longer than that of a circle and, therefore, squeezing occurs under less restraint. The pressure on a strip is greater than on a circle, if $r_0 = b_0$, because now the perimeter per area of a circle is greater than that of a strip. For comparison, Fig. 1 depicts curves of pressure $p_0 = Ph_0^2/\rho_0 w_0^2 b_0^2$ at the center of a strip (curve 1) and at the center of a circle (curve 2) during squeezing at constant velocity.

In experiments β is usually large and, therefore, it is reasonable to assume a constant squeezing velocity. A solution based on this assumption is exact, moreover, inasmuch as the combination of terms $dw/\partial t + w\partial w/dz = 0$ in (1.1) when $w_0 =$ const, and thus the axial projection of the Euler equation is satisfied exactly rather than approximately even when h/b is small.

Let the magnitude of ε be arbitrary but the mass of the punch be large. Solution (2.9) can then be found by expanding into a series in the small parameter β^{-1}, and already the zeroth-order term in ψ_0 satisfying the equation

$$2\xi\frac{d\psi_0}{d\xi} = \delta\psi_0^2 + 2(1+\varepsilon)\psi_0 - 4\varepsilon, \quad \psi_0(1) = \varepsilon \qquad (2.17)$$

will at $\beta \to \infty$ give a satisfactory approximation in the $10\beta^{-1} \le \xi \le 1$ range. Integrating (2.17) will yield

$$\delta\psi_0(\alpha + \xi^k) = (\delta - 2 - k)\xi^k + \alpha(k - 2 + \delta), \quad \alpha(k - 2 + \delta^2) = k + 2 - \delta^2$$
$$p = \frac{\rho a_0 \varepsilon}{8}\left(\frac{w_0^\circ}{h_0\xi}\right)^2\left[(1-\psi_0)^2 + 3\right]\left[1 - \frac{x^2}{a_0^2} - \frac{y^2}{b_0^2}\right], \quad \theta = 1/\xi, \quad \tau = 1 - \xi$$

The position of an $\varepsilon \ne 1$ node does not coincide with (2.12) in the zeroth approximation, i.e., when $\xi \to 0$, $\psi_0 \to \psi_*$, and $\psi_* = 1 - (2-k)/\delta \le 1$, and a small difference in ψ_0 from the exact solution appears when ξ is small. When $\varepsilon = 1$, however, the approximate solution becomes exact because $\psi_0 \equiv 1$ now.

The graphs of functions $\psi(\xi)$ and $\tau(\xi)$ are shown in Fig. 2 and in Fig. 3 for $\varepsilon = 0.5$, $\beta = 10$, 100, and 1000 (corresponding curves 1, 2, and 3) and, for comparison, the zeroth approximation has been indicated by a dashed line. It is evident here that a difference between the approximate and the exact solution becomes noticeable only at $\xi \le 10\beta^{-1}$.

It would be interesting to compare the spreading velocities v_b and u_a along the minor and the major semiaxis of an ellipse,

$$\frac{v_b}{u_a}=\frac{U}{V}\left(\frac{\varepsilon}{2-\varepsilon}\right)^{1/2},\quad \frac{v}{u}=\frac{y}{x}\frac{V}{U},\quad \frac{U}{V}=\frac{2-\psi}{\psi} \tag{2.18}$$

Initially $\psi=\varepsilon<1$ and, according to (2.18), $v_b>u_a$. This fact is explained by the large pressure gradient along the minor semiaxis, which results in a rotation of the velocity vector $v/u \geq y/x$. During squeezing at a constant velocity, $\tau \to 1$ and $\psi \to 1-(2-k)/\delta$, and thus always $v_b>u_a$.

If the mass of the punch is finite, then $\psi \to 1$ when $\xi \to 0$ and, therefore, $v_b<u_a$. This fact has to do with the inertia of the spreading liquid when the pressure forces are negligible.

We will now consider the exact solution for the case where the film is squeezed by a constant external force $f_0=\pi a_0^3 b_0 \rho\,(U^2+\dot{U})/4$. Integrating this expression and using (2.2) will yield

$$\frac{A+U}{A-U}=\frac{A+U_0}{A-U_0}\exp 2At,\quad \frac{B+V}{B-V}=\frac{B+V_0}{B-V_0}\exp 2Bt$$
$$A=4f_0/\pi\rho a_0^3 b_0,\quad B=a_0A/b_0$$

We note that Eqs. (2.2) are simply integrated also when $\varepsilon \to 0$ and $f_0/a_0=\text{const}$. The spreading of a liquid between two flat surfaces is described by the same equations (2.2) and (2.3), if one considers that $\varepsilon=\varepsilon(t)$ and $\beta=\beta(t)$, since

$$da/dt=aU,\quad abh=a_0b_0h_0,\quad a(0)=a_0 \tag{2.19}$$

An elliptical film remains elliptical when hammered out. Indeed, any arbitrary point on an ellipse with semiaxes a_1, b_1 whose coordinates at time t_1 are x, y will after the time t_1+dt have moved to the point whose coordinates are

$$x_1=x_1+Ux_1dt_1,\quad y_1=y_1+Vy_1\,dt$$

and which belongs to the ellipse with semiaxes

$$a_2=a_1+Ua_1dt,\qquad b_2=b_1+Vb_1dt$$

This can be verified by breaking up the equation $x^2_2/a_2^2+y^2_2/b^2_2=1$ with an accuracy of dt and by considering that

$$x^2_1/a^2_1+y^2_1/b^2_1=1$$

Of practical interest is the impact of a large mass, when $w_0^\circ=\text{const}$ and (2.2), (2.3), and (2.19) will yield

$$2\xi\frac{d\varphi}{d\xi}=4\varphi-\delta(\varphi^2+3),\quad \xi\frac{d\delta}{d\xi}=\varphi(1-\delta^2),\quad 2\xi\frac{d\bar{a}}{d\xi}=\bar{a}(\varphi-1)$$
$$\varphi(1)=\delta_0,\quad \bar{a}(1)=1,\quad \bar{a}=a/a_0,\quad \delta(1)=\delta_0 \tag{2.20}$$

From the first two equations in (2.20) we get the Abel equation:

$$2\varphi(1-\delta^2)\frac{d\varphi}{d\delta}=4\varphi-\delta(\varphi^2+3)$$

This equation has the following singular points: focus (0,0), saddle points (1,1) and (−1,1), and nodes (1,3) and (−1, −3). The vertical lines $\delta=\pm 1$ represent particular solutions with the integral curves approached as the derivative becomes infinite. The zero isoclinic $\delta=4\varphi/(\varphi^2+3)$, shown by a dashed line in Fig. 4 originates at the singular point (1,1) has a slope $d\varphi/d\delta=2$, and lies below the separatrix $\varphi=1+{}^2/_3(\delta-1)$. The initial conditions (point 1) lie on the bisector $\varphi=\delta$ indicated by a dashed-dotted line.

It becomes evident from Fig. 4 representing the field of integral curves that under continuous squeezing an ellipse transforms into a circle with some radius r_2 (point 2). At that time, however, the velocity field still remains asymmetrical, because

$$a=b=r_2,\quad U/V=(1-\varphi)/(1+\varphi)<1,$$

when $\varphi=\varphi_2<1$, i.e., $v_b>u_a$, and thus the circle transforms further into another ellipse – with semiaxes $b>a$ this time (segment between points 2 and 3 on the integral curve in Fig. 4). This process is repeated with the eccentricity decreasing every time and with the focal distance $|a^2-b^2|^{1/2}$ oscillating.

If Eqs. (2.20), in the vicinity of a focus, are written in polar coordinates

$$2\,\frac{d\omega}{d\eta}=2+\cos^2\omega-2\sin 2\omega+0.25r^2\sin^2 2\omega,\quad \eta=-\ln\xi,\quad \varphi=r\sin\omega,$$
$$\delta=r\cos\omega,\quad 0\leqslant r\leqslant 1$$

then it becomes evident that, as $h\to 0$ $(\eta\to\infty)$, $\omega\to\infty$ because $d\omega/d\eta>0$; i.e., the number of oscillations is infinite. It is to be noted that the separatrix originating at point (1,1) (heavy line in Fig. 4) may not be treated as the solution to the problem, since the initial condition $\delta_0=1$ corresponds to the degeneration of an ellipse into a strip $\varphi\equiv 1$. However, one may approach the separatrix arbitrarily close and the oscillations will start only when the squeezing is almost complete, i.e., when $h\to 0$. With a and b expressed, according to (2.20), as

$$a=a_0\xi^{-1/2}\exp\frac{1}{2}\int\limits_1^{\xi}\frac{\varphi}{\xi}\,d\xi,\quad b=b_0\xi^{-1/2}\exp-\frac{1}{2}\int\limits_1^{\xi}\frac{\varphi}{\xi}\,d\xi$$

we will find that the amplitudes $K=(a-b)/a_0$ of the ellipse axes oscillation are extremal K_* at points ξ_0 representing the roots of the equation

$$[1-\varphi(\xi_0)]\exp\int\limits_1^{\xi_0}\frac{\varphi}{\xi}\,d\xi=1+\varphi(\xi_0),\quad K_*=2\left(\frac{b_0}{a_0\xi_0}\right)^{1/2}\frac{\varphi(\xi_0)}{1+\varphi(\xi_0)}$$

It follows from (2.20) that $\delta\to 0$ when $\xi\to 0$, in accordance with the equation

$$\frac{d^2\delta}{d\eta^2}+2\frac{d\delta}{d\eta}+\frac{3}{2}\,\delta=0,\quad \varphi\to\frac{d\delta}{d\eta}$$
$$\delta\to\xi\left(A\sin\frac{\ln\xi}{\sqrt{2}}+B\cos\frac{\ln\xi}{\sqrt{2}}\right),\quad \varphi\to\xi\left(A_1\sin\frac{\ln\xi}{\sqrt{2}}+B_1\cos\frac{\ln\xi}{\sqrt{2}}\right)$$
$$K\to\xi^{1/2}\left(A_2\sin\frac{\ln\xi}{\sqrt{2}}+B_2\cos\frac{\ln\xi}{\sqrt{2}}\right)$$

For this reason, the amplitudes of oscillation decrease, and an ellipse becomes a circle. The curve in Fig. 5 represents the relative film thicknesses ξ_* at which an ellipse becomes a circle for the first time (point 2 in Fig. 4), calculated as a function of δ_0. It is noticeable that the shape of an ellipse starts to oscillate later when ξ is small. And only when the ellipse is almost a circle at the very beginning will oscillations occur already at light squeezing. Thus, for example, calculations have shown that ξ_* is equal to 0.16 or 0.29 respectively when $\delta_0=10^{-3}$ or 10^{-4}. The curves in Fig. 6a show the variation in the semi-axes of an ellipse, $\bar{a}$ and $\bar{b}=b/a_0$, calculated for $\delta_0=0.43$. The curves in Fig. 6b represent $K=\bar{a}-\bar{b}$ calculated for $\delta_0=0.43$ and 0.55. For comparison with theory (É. I. Andriankin) we also show the test curves in Fig. 6a, b which have been obtained by V. K. Bobolev and A. V. Dubovik: (1) for aqueous glycerine, $\rho=1.24$ g/cm^3, $\mu=3$ poise, $w_0=2$ m/sec, $m_0=5$ kg, punch radius 9.5 mm, $h_0=0.25$ mm, $a_0=4.45$ mm, $b_0=2.85$ mm, Re = 1, $\delta_0=0.42$, and $\beta=1.76\cdot 10^4$; (2) for $h_0=0.5$ mm, $a_0=0.45$ mm, $b_0=2.85$ mm, Re = 2, $\delta_0=0.43$, and $\beta=3.5\cdot 10^4$; (3) for honey, $\rho=1.41$ g/cm^3, $\mu=100$ poise, $w_0=1$ m/sec, $m_0=10$ kg, $h_0=0.5$ mm, $a_0=4.75$ mm, $b_0=2.55$ mm, Re ≈ 0.35, $\delta_0=0.55$, and $\beta=7.4\cdot 10^4$.

The problem of spreading is simple to solve in the case of a circular film $a=b=R(t)$, even when the punch deceleration is taken into account, because then

$$U=V,\quad \beta=\beta_0\xi^2,\quad \psi=1,\quad \beta_0=8h_0m_0/\pi\rho r_0^4$$

Integrating (2.10) and (2.11) yields with (2.19)

$$R=r_0\xi^{-1/2},\quad U=-\frac{w_0^\circ}{2h_0}\theta,\quad \theta^2=\frac{\xi(1+\beta_0)}{1+\beta_0\xi^3}$$
$$\tau=2(\xi^{-1/2}-1)+0.4\beta_0(1-\xi^{5/2})$$

If the punch is stopped at time t_1 as a result of some external action whatsoever, then the liquid will continue to spread due to inertia and

$$p=0,\quad \dot{U}+U^2=0,\quad UR_*=dR_*/dt,\quad R_1{}^2h_1=R_*{}^2h$$
$$U=U_1R_1/R_*,\quad R_*=R_1[1+U_1(t-t_1)]$$

3. Slow Spreading of a Viscous Film. Disregarding the inertial terms in (1.4) will leave the following equations:

$$a^2 \frac{\partial^2 U}{\partial z^2} = b^2 \frac{\partial^2 V}{\partial z^2}, \quad U + V + \frac{\partial w}{\partial z} = 0, \quad m_0 \frac{dw^\circ}{dt} = \frac{\pi ab}{2} P(t)$$
$$P(t) = -\frac{\mu a^2}{2} \frac{\partial^2 U}{\partial z^2}$$

Integrating them with the conditions of adhesion taken into account, we have

$$U = z(\delta - z) U^\circ(t), \quad V = z(\delta - z) V^\circ(t), \quad w = -\frac{z^2(3h - 2z)}{3\varepsilon} U^\circ(t)$$
$$P = \mu a^2 U^\circ, \quad V^\circ = a^2 U^\circ / b^2, \quad U^\circ = -3\varepsilon h^{-3} w^\circ(t)$$

Integrating the equation of motion for the punch yields

$$w^\circ(t) = w_0^\circ + \frac{3\pi\mu a_0^3 b_0^3 (1 - \xi^2)}{2m_0 h_0^2 \xi^2 (a_0^2 + b_0^2)} \tag{3.1}$$
$$\tau = (1 - \xi_k)\left[1 - \xi - \frac{\xi_k}{2} \ln \frac{(\xi - \xi_k)(1 + \xi_k)}{(\xi + \xi_k)(1 - \xi_k)}\right]$$
$$\xi_k^{-2} = 1 + \frac{2m_0 h_0^2 w_0^\circ (a_0^2 + b_0^2)}{3\pi \mu a_0^3 b_0^3}$$

It follows from (3.1) that the punch comes to a stop within a finite film thickness ξ_f, but the time of motion until full stop – proportional to $\ln(\xi - \xi_k)$ – is infinite. If we compare the finite film thicknesses ξ_f° and ξ_f under impact with a circular and an elliptical punch, respectively, both having the same mass and base area $r_0^2 = a_0 b_0$ as well as the same velocity, then we arrive at the entirely obvious result that $\xi_k^0 > \xi_k$; i.e., that a narrow film is more easily squeezed.

In solving the problem of an elliptical film hammered out between two flat plates, we will assume that the ellipse boundary expands at the mean-over-the-thickness velocity. Then

$$a^\cdot = \frac{U^\circ a}{h} \int_0^h z(h - z)\, dz = \frac{U^\circ a h^2}{6}, \quad b^\cdot = \frac{V^\circ b h^2}{6} \tag{3.2}$$

These relations satisfy the condition of constant liquid mass $abh = a_0 b_0 h_0$.

Considering that $V^\circ = a^2 U^2 b^{-2}$, we have from (3.2)

$$a^2 - b^2 = c_0^2, \quad c_0^2 = a_0^2 - b_0^2, \quad a^2 = \frac{c_0^2}{2} + \lambda^{1/2}, \quad \lambda = \frac{c_0^4}{4} + \left(\frac{a_0 b_0}{\xi}\right)^2 \tag{3.3}$$
$$K = \left(\frac{\delta_0}{1 + \delta_0}\right)^{1/2} [(1 + \omega_1)^{1/2} - (\omega_1 - 1)^{1/2}], \quad \omega_1^2 = 1 + \frac{1 - \delta_0^2}{\delta_0^2 \xi^2}$$

i.e., the focal distance is maintained (unlike in the inertial case), but the difference $(a - b)/a$ decreases monotonically. The function $K(\xi)$ has been plotted in Fig. 6b for $\delta_0 = 0.43$ and 0.55. However, the finite mass of the punch will come to a stop when $\xi_k > 0$ and, therefore, the ellipse does not transform into a circle. Indeed, integrating the equation of motion for the punch and taking into consideration (3.3), we obtain

$$w^\circ(\xi) = w_0^\circ + \frac{3\pi\mu a_0^2 b_0^2}{8m_0 h_0^2} \left[\xi^{-4} - 1 + \frac{2c_0^2(\lambda^{3/2} - \lambda_1^{3/2})}{3a_0^4 b_0^4} - \frac{c_0^6(\lambda^{1/2} - \lambda_1^{1/2})}{2a_0^4 b_0^4}\right]$$
$$4\lambda_1 = a_0^2 + b_0^2$$

Since the first term inside the brackets increases fast when $\xi \to 0$, hence $w^\circ(\xi)$ vanishes when $\xi_k \to 0$. It is also not surprising that the curves a and b representing the inertial and the viscous case, respectively, are close (points 4 in Fig. 6a), since the spreading is governed by the mass balance, even though the pressures are quite different.

In conclusion, we write the equation of motion for a circular film of liquid squeezed under a constant force $f_0 = \pi r_0^2 P_0 / 2$ and include here both the inertial and the viscous term.

The solution will be sought in the form

$$U \equiv V, \quad U = -\frac{1}{2} \frac{\partial w}{\partial z}, \quad w = w(z)$$

From (1.4) we then obtain an equation for w which is analogous to the boundary-layer equation,

$$2\nu \frac{d^3 w}{dz^3} - 2w \frac{d^2 w}{dz^2} + \left(\frac{dw}{dz}\right)^2 = \alpha^2, \quad \alpha^2 = \frac{16 f_0}{\pi r_0^4 \rho}$$

and which can be solved in the form of a series.

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